Discretization

Continuous world
- Surface $X$
- Metric $d_X$
- Topology

Discrete world
- Sampling $X' = \{ x_1, ..., x_n \} \subset X$
- Discrete metric (matrix of distances) $D_X = (d_X(x_i, x_j))$
- Discrete topology (connectivity)

How good is a sampling?

Sampling density
- How to quantify density of sampling?
- $X'$ is an $r$-covering of $X$ if $\bigcup_{x_i \in X'} B_r(x_i) = X$
- Alternatively:
  \[ d_X(x, X') \leq r \]
  for all $x \in X$, where
  \[ d_X(x, x') = \inf_{x_i \in X'} d_X(x, x_i) \]
  is the point-to-set distance.

Sampling efficiency
- Are all points necessary?
- An $r$-covering may be unnecessarily dense (may even not be a discrete set).
- Quantify how well the samples are separated.
- $X'$ is $r'$-separated if $d_X(x_i, x_j) \geq r'$
  for all $x_i, x_j \in X$.
- For $r' > 0$ an $r'$-separated set is finite if $X$ is compact.

Farthest point sampling
- Good sampling has to be dense and efficient at the same time.
- Find and $r$-separated $r'$-covering $X'$ of $X$.
- Achieved using farthest point sampling.
- We defer the discussion on:
  - How to select $r$?
  - How to compute $d_X$?
Farthest point sampling

- Start with some \( X' = \{ x_1 \in X \} \).
- Determine sampling radius
  \[ r = \max_{x \in X} d_X(x, X') \]
- If \( r \leq \text{target} \) stop.
- Find the farthest point from \( X' \)
  \[ x' = \arg \max_{x \in X} d_X(x, X') \]
- Add \( x' \) to \( X' \).

Outcome: \( r \)-separated \( r \)-covering of \( X \).
Produces sampling with progressively increasing density.
A greedy algorithm: previously added points remain in \( X' \).
There might be another \( r \)-separated \( r \)-covering containing less points.
In practice used to sub-sample a densely sampled shape.
Straightforward time complexity: \( O(MN) \)
  - \( M \) number of points in dense sampling.
  - \( N \) number of points in \( X' \).
Using efficient data structures can be reduced to \( O(N \log M) \).

Sampling as representation
- Sampling represents a region on \( X \) as a single point \( x_i \in X' \).
- Region of points on \( X \) closer to \( x_i \) than to any other \( x_j \)
  \[ V_i(X') = \{ x \in X : d_X(x, x_i) < d_X(x, x_j), x_j \neq i \in X' \} \]
- Voronoi region (a.k.a. Dirichlet or Voronoi-Diichlet region, Thiessen polytope or polygon, Wigner-Seitz zone, domain of action).
- To avoid degenerate cases, assume points in \( X' \) in general position:
  - No three points lie on the same geodesic.
    (Euclidean case: no three collinear points).
  - No four points lie on the boundary of the same metric ball.
    (Euclidean case: no four cocircular points).

A point \( x \in X \) can belong to one of the following
- Voronoi region \( V_i \) ( \( x \) is closer to \( x_i \) than to any other \( x_j \)).
- Voronoi edge \( V_{ij} = V_i \cap V_j \) ( \( x \) is equidistant from \( x_i \) and \( x_j \)).
- Voronoi vertex \( V_{ijk} = V_i \cap V_j \cap V_k \) ( \( x \) is equidistant from three points \( x_i, x_j, x_k \)).
Voronoi decomposition
- Voronoi regions are disjoint.
- Their closure \( \bigcup V_i = X \) covers the entire \( X \).
- Cutting \( X \) along Voronoi edges produces a collection of tiles \( \{V_i\} \).
- In the Euclidean case, the tiles are convex polygons.
- Hence, the tiles are topological disks (are homeomorphic to a disk).

Tessellation of \( X \) (a.k.a. cell complex):
- a finite collection of disjoint open topological disks, whose closure cover the entire \( X \).
- In the Euclidean case, Voronoi decomposition is always a tessellation.
- In the general case, Voronoi regions might not be topological disks.
- A valid tessellation is obtained if the sampling \( X' \) is sufficiently dense.

Non-Euclidean Voronoi tessellations
- Convexity radius at a point \( x \in X \) is the largest \( \rho \) for which the closed ball \( B_\rho(x) \) is convex in \( X \), i.e., minimal geodesics between every \( x', x'' \in B_\rho(x) \) lie in \( B_\rho(x) \).
- Convexity radius of \( X \) = infimum of convexity radii over all \( x \in X \).
- Theorem (Leibon & Letscher, 2000):
  An \( r \)-separated \( r \)-covering \( X' \) of \( X \) with \( r < \frac{1}{3} \) convexity radius of \( X' \) is guaranteed to produce a valid Voronoi tessellation.
- Gives sufficient sampling density conditions.

Sufficient sampling density conditions
- Invalid tessellation
- Valid tessellation

MATLAB® intermezzo
Farthest point sampling and Voronoi decomposition
**Representation error**

- Voronoi decomposition replaces \( x \in X \) with the closest point \( x^* \in X' \).
- Mapping \( x^* : X \to X' \) copying each \( V_i(X') \) into \( x_i \).
- Quantify the representation error introduced by \( x^* \).
- Let \( x \in X \) be picked randomly with uniform distribution on \( X \).

\[
P(x \in A) = \mu(A) \mu(X) = \frac{1}{\mu(X)} \int_A dx
\]

- Representation error = variance of \( d_X(x, x^*(x)) \)

\[
\epsilon(X') = \text{Var}(d_X(x, x^*(x))) = \frac{1}{\mu(X)} \int_{x \in X} \left( d_X(x, x^*(x)) \right)^2 dx
\]

**Optimal sampling**

- In the Euclidean case:

\[
\epsilon(X') = \frac{1}{\mu(X)} \sum_{i=1}^{N} \int_{x \in X} \|x - x_i\|^2 dx
\]

(mean squared error).

- Optimal sampling: given a fixed sampling size \( N \), minimize error

\[
X' = \arg\min_{X'} \epsilon(X') \text{ s.t. } \|X'\| = N
\]

Alternatively: Given a fixed representation error \( \epsilon_0 \), minimize sampling size

\[
X' = \arg\min_{X'} \|X'\| \text{ s.t. } \epsilon(X') \leq \epsilon_0
\]

**Centroidal Voronoi tessellation**

- In a sampling \( X' \) minimizing \( \epsilon(X') \), each \( x_i \) has to satisfy

\[
\begin{align*}
  x_i &= \arg\min_{x \in V_i} \int_{x^* \in V_i} \|x - x^*\|^2 dx^* \\
  &= \text{(intrinsic centroid)}
\end{align*}
\]

- In the Euclidean case – center of mass

\[
x_i = \arg\min_{x \in V_i} \int_{x^* \in V_i} \|x - x^*\|^2 dx^* = \frac{\int_{V_i} x dx}{|V_i|}
\]

- In general case: Intrinsic centroid of \( V_i \).
- Centroidal Voronoi tessellation (CVT): Voronoi tessellation generated by \( X' \) in which each \( x_i \) is the intrinsic centroid of \( V_i(X') \).

**Lloyd-Max algorithm**

- Start with some sampling \( X' \) (e.g., produced by FPS)
- Construct Voronoi decomposition \( \{V_i(X')\} \)
- For each \( i \), compute intrinsic centroids

\[
x_i = \arg\min_{x \in V_i} \int_{x^* \in V_i} d_X(x, x') dx'
\]

- Update \( X' = \{x_1, \ldots, x_N\} \)
- In the limit \( N \to \infty \), \( \{V_i(X')\} \) approaches the hexagonal honeycomb shape – the densest possible tessellation.
- Lloyd-Max algorithms is known under many other names: vector quantization, k-means, etc.

**Farthest point sampling encore**

- Start with some \( x_1 \in X \), \( R_1 = \infty \)
- For \( i = 2, \ldots, N \)
- Find the farthest point

\[
x_i = \arg\max_{x \in X} d_X(x, \{x_1, \ldots, x_{i-1}\})
\]

- Compute the sampling radius

\[
R_i = d_X(x_i, \{x_1, \ldots, x_{i-1}\})
\]

Lloyd-Max algorithm, a.k.a. k-means is a heuristic, sometimes minimizing average cluster radius \( \epsilon_0 \) (if converges globally – not guaranteed)
Almost optimal sampling

Theorem (Hochbaum & Shmoys, 1985)
Let \( x_1, \ldots, x_N \) be the result of the FPS algorithm. Then
\[
\epsilon_{w}(\{x_1, \ldots, x_N\}) \leq 2 \min \epsilon_{w}
\]
In other words: FPS is worse than optimal sampling by at most 2.

Connectivity

- Neighborhood is a topological definition independent of a metric
- Two points are adjacent (directly connected) if they belong to the same neighborhood
- The connectivity structure can be represented as an undirected graph with vertices \( V = \{1, \ldots, N\} \) and edges \( E = \{(i,j) \mid i \times j \in N(x_i)\} \)
- Connectivity graph can be represented as a matrix
\[
\mathbf{e}_{ij} = \begin{cases} 
1 & x_i \text{ and } x_j \text{ connected} \\
0 & \text{else}
\end{cases}
\]

Shape representation

Cloud of points
Graph

Delaunay tessellation

Define connectivity as follows: a pair of points whose Voronoi cells are adjacent are connected
The obtained connectivity graph is dual to the Voronoi diagram and is called Delaunay tessellation

Delaunay tessellation

For surfaces, the triangles are replaced by geodesic triangles
[Leibon & Letscher]: under conditions that guarantee the existence of Voronoi tessellation, Delaunay triangles form a valid tessellation
Replacing geodesic triangles by planar ones gives Delaunay triangulation
Shape representation

Cloud of points

Graph

Triangular mesh

Triangular meshes

The geometric realization of the mesh is defined by specifying the coordinates of the vertices $x_i \in \mathbb{R}^3$ for all $i \in \mathcal{I}$.

The coordinates can be represented as an $N \times 3$ matrix.

The mesh is a piece-wise planar approximation obtained by gluing the triangular faces together,

$$T(\mathcal{X}) = \bigcup_{k=1}^{N_F} \text{conv}(x_{k1}, x_{k2}, x_{k3})$$

Example of a triangular mesh

Vertices

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tr>
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<td>1, 3</td>
<td>1, 4</td>
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<td>4, 2</td>
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Faces

The geometric realization of the mesh is defined by specifying the coordinates of the vertices $x_i \in \mathbb{R}^3$ for all $i \in \mathcal{I}$.

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The mesh is a piece-wise planar approximation obtained by gluing the triangular faces together,

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Example of a triangular mesh

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Faces

Barycentric coordinates

Any point on the mesh $T(\mathcal{X})$ can be represented providing

- index $k$ of the triangle enclosing it;
- coefficients of the convex combination of the triangle vertices

$$z = u_1 x_{k1} + u_2 x_{k2} + (1 - u_1 - u_2) x_{k3} ; \ u_i \in [0, 1]$$

Vector $u = (u_1, u_2)$ is called barycentric coordinates.

Manifold meshes

- $T(\mathcal{X})$ is a manifold
- Neighborhood of each interior vertex is homeomorphic to a disc
- Neighborhood of each boundary vertex is homeomorphic to a half-disc
- Each interior edge belongs to two triangles
- Each boundary edge belongs to one triangle
Non-manifold meshes

Edge shared by four triangles

Non-manifold mesh

Geometry images

Surface
\( \{u, v, z(u, v)\} \)

Geometry image
\( z_{ij} = z(u_{ij}, v_{ij}) \)

Global parametrization \( \{x(u, v), y(u, v), z(u, v)\} \)

Sampling of parametrization domain on a Cartesian grid
\( (u_{ij}, v_{ij}) = (i\Delta u, j\Delta v) \)

Geometry images

Six-neighbor

Eight-neighbor

Manifold mesh

Non-manifold mesh

Geometric validity

Topologically valid

Geometrically invalid

Topological validity (manifold mesh) is insufficient!

Geometric validity means that the realization of the triangular mesh does not contain self-intersections

Skeleton

For a smooth compact surface \( X \), there exists an \textit{envelope} \( V_X \) (open set in \( \mathbb{R}^3 \) containing \( X \)) such that every point \( u \in V_X \) is continuously mappable to a unique point on \( X \)

The mapping is realized as the closest point on \( X \) from \( u \)

Problem when \( u \) is \textit{equidistant} from two points on \( X \)
(such points are called \textit{medial axis} or \textit{skeleton} of \( X \))

If the mesh \( T(X') \) is contained in the envelope \( V_X \)
does not intersect the medial axis), it is valid

Skeleton

Points equidistant from the boundary form the skeleton of a shape
Local feature size

Distance from point \( x \) on \( X \) to the medial axis of \( X \) is called the local feature size, denoted \( \rho(x) \).

Local feature size related to curvature \( \rho(x) \leq \frac{1}{\max\{\kappa_1(x), \kappa_2(x)\}} \) (not an intrinsic property!)

[Amenta&Bern, Leibon&Letscher]: if the surface is sampled such that for every \( x \) an open ball of radius \( \frac{1}{4} \rho(x) \) contains a point of \( X \) it is guaranteed that \( T(X^o) \) does not intersect the medial axis of \( X \).

Conclusion: there exists sufficiently dense sampling guaranteeing that \( T(X^o) \) is geometrically valid.